The Jacobian of SVD

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1 Problem Description

As we know, for any matrix A, we have such a formula:

$$A = U\Sigma V^T$$

An interesting question then comes up with this formula: If I did a derivative of the SVD decomposition, what can I get?

Actually, this question is crucial for the simulation, which has already been well discussed in the Siggraph course DynamicDeformables[KE20]. For instance, when we want the As-Rigid-As-Possible energy, it looks like this:

$$\phi(F) = \|F - R\|_F^2, F = RS$$

In this energy formula, F is the Jacobian Deformation Matrix and it is composed of the rotation matrix R and scale matrix S.

Okay, then how could we get the Jacobian matrix of this energy function, to get the force? It is natural that we first rewrite this formula:

$$\begin{aligned}
\frac{\partial \phi}{\partial f_{ij}} &= \frac{\partial(tr(F^TF) - 2tr(R^TF) + tr(R^TR))}{\partial f_{ij}} \\
\frac{\partial \phi}{\partial f_{ij}} &= \frac{\partial(tr(F^TF) - 2tr(S) + tr(I))}{\partial f_{ij}} \ (F = RS\text{,and } R^TR = I \text{ since } R \text{ is a rotation matrix}) \\
\frac{\partial \phi}{\partial f_{ij}} &= \frac{\partial(tr(F^TF) - 2tr(R^TF) + 3)}{\partial f_{ij}} \\
\frac{\partial \phi}{\partial f_{ij}} &= 2(F - R)
\end{aligned}$$

For now, everything looks good. But how about I want to compute the Hessian matrix of the energy function? Then I need to know what is $\frac{\partial R}{\partial f_{ij}}$, which is also written as $\frac{\partial R}{\partial F}$. This question is also called estimating Jacobian of SVD decomposition matrix[PL00].

2 The Derivation of SVD's Jacobian matrix

Let's differentiate the SVD function directly:

$$\begin{aligned}
F &= U\Sigma V^T \\
\frac{\partial F}{\partial f_{ij}} &= \frac{\partial U}{\partial f_{ij}}\Sigma V^T + U\frac{\partial \Sigma}{\partial f_{ij}}V^T + U\Sigma(\frac{\partial V}{\partial f_{ij}})^T \\
U^T\frac{\partial F}{\partial f_{ij}}V &= U^T\frac{\partial U}{\partial f_{ij}}\Sigma + \frac{\partial \Sigma}{\partial f_{ij}} + \Sigma(\frac{\partial V}{\partial f_{ij}})^TV
\end{aligned}$$

We can take advantage of the fact that both U and V are orthogonal matrices here. Therefore, they we can do this(just do it for U first):

$$\frac{\partial U^T U}{\partial f_{ij}} = (\frac{\partial U}{\partial f_{ij}})^T U + U^T \frac{\partial U}{\partial f_{ij}}$$

Since $U^T U = I$, this equation should equal to 0. That is:

$$(\frac{\partial U}{\partial f_{ij}})^T U + U^T \frac{\partial U}{\partial f_{ij}} = 0$$

Also, we noticed that:

$$((\frac{\partial U}{\partial f_{ij}})^T U)^T = U^T \frac{\partial U}{\partial f_{ij}}$$

So $(\frac{\partial U}{\partial f_{ij}})^T U$ is an anti-symmetric matrix. And the same for $V^T \frac{\partial V}{\partial f_{ij}}$, it should be an anti-matrix too.

Let me try to figure out what these two matrices look like(u_i and v_i are the variables wait to be solved):

$$(\frac{\partial U}{\partial f_{ij}})^T U = \begin{bmatrix} 0 & u_1 & u_2 \\ -u_1 & 0 & u_3 \\ -u_2 & -u_3 & 0 \end{bmatrix}, V^T \frac{\partial V}{\partial f_{ij}} = \begin{bmatrix} 0 & v_1 & v_2 \\ -v_1 & 0 & v_3 \\ -v_2 & -v_3 & 0 \end{bmatrix}$$

$\frac{\partial \Sigma}{\partial f_{ij}}$ is a diagonal matrix, which means we can safely say we don't care what it is before we solve the sum of two anti-symmetric matrices. What a coincidence! Then we have:

$$\begin{aligned} U^T \frac{\partial F}{\partial f_{ij}} V &= U^T \frac{\partial U}{\partial f_{ij}} \Sigma + \frac{\partial \Sigma}{\partial f_{ij}} + \Sigma (\frac{\partial V}{\partial f_{ij}})^T V \\ U^T \frac{\partial F}{\partial f_{ij}} V - \frac{\partial \Sigma}{\partial f_{ij}} &= U^T \frac{\partial U}{\partial f_{ij}} \Sigma + \Sigma (\frac{\partial V}{\partial f_{ij}})^T V \\ U^T V - \frac{\partial \Sigma}{\partial f_{ij}} &= U^T \frac{\partial U}{\partial f_{ij}} \Sigma + \Sigma (\frac{\partial V}{\partial f_{ij}})^T V \quad \textbf{(Because } \frac{\partial F}{\partial f_{ij}} = I\textbf{)} \end{aligned}$$

With known F, we can compute U and V(let's just fill the matrix $U^T V$ with a_{ij}, and $\frac{\partial \Sigma}{\partial f_{ij}}$ with x_i since we don't care about it currently), so for the left side, it looks like this:

$$U^T V - \frac{\partial \Sigma}{\partial f_{ij}} = \begin{bmatrix} a_{00} & a_{01} & a_{02} \\ a_{10} & a_{11} & a_{12} \\ a_{20} & a_{21} & a_{22} \end{bmatrix} - \begin{bmatrix} x_0 & 0 & 0 \\ 0 & x_1 & 0 \\ 0 & 0 & x_2 \end{bmatrix} = \begin{bmatrix} a_{00} - x_0 & a_{01} & a_{02} \\ a_{10} & a_{11} - x_1 & a_{12} \\ a_{20} & a_{21} & a_{22} - x_2 \end{bmatrix}$$

Let's say the Σ matrix is:

$$\Sigma = \begin{bmatrix} \sigma_0 & 0 & 0 \\ 0 & \sigma_1 & 0 \\ 0 & 0 & \sigma_2 \end{bmatrix}$$

For the right side, it looks like:

$$\begin{aligned} U^T \frac{\partial U}{\partial f_{ij}} \Sigma + \Sigma (\frac{\partial V}{\partial f_{ij}})^T V &= \begin{bmatrix} 0 & \sigma_1 u_1 & \sigma_2 u_2 \\ -\sigma_0 u_1 & 0 & \sigma_2 u_3 \\ -\sigma_0 u_2 & -\sigma_1 u_3 & 0 \end{bmatrix} + \begin{bmatrix} 0 & \sigma_0 v_1 & \sigma_0 v_2 \\ -\sigma_1 v_1 & 0 & \sigma_1 v_3 \\ -\sigma_2 v_2 & -\sigma_2 v_3 & 0 \end{bmatrix} \\ &= \begin{bmatrix} 0 & \sigma_1 u_1 + \sigma_0 v_1 & \sigma_2 u_2 + \sigma_0 v_2 \\ -(\sigma_0 u_1 + \sigma_1 v_1) & 0 & \sigma_2 u_3 + \sigma_1 v_3 \\ -(\sigma_0 u_2 + \sigma_2 v_2) & -(\sigma_1 u_3 + \sigma_2 v_3) & 0 \end{bmatrix} \end{aligned}$$

And for now, we just need to solve three 2 by 2 linear systems.
then we have:

$$\begin{cases} \sigma_1 u_1 + \sigma_0 v_1 = a_{01} \\ -\sigma_0 u_1 + \sigma_1 v_1 = a_{10} \end{cases}, \begin{cases} \sigma_2 u_2 + \sigma_0 v_2 = a_{02} \\ -\sigma_0 u_2 + \sigma_2 v_2 = a_{20} \end{cases}, \begin{cases} \sigma_2 u_3 + \sigma_1 v_3 = a_{12} \\ -\sigma_1 u_3 + \sigma_2 v_3 = a_{21} \end{cases}$$

When we have this, we solved $\frac{\partial U}{\partial f_{ij}}$ and $\frac{\partial V}{\partial f_{ij}}$, which means we exactly know the matrix $U^T \frac{\partial U}{\partial f_{ij}} \Sigma + \Sigma(\frac{\partial V}{\partial f_{ij}})^T V$. Let's call it S. Then the $\frac{\partial \Sigma}{\partial f_{ij}}$ is easy to compute:

$$\frac{\partial \Sigma}{\partial f_{ij}} = S - U^T V$$

3 Back to the APAR Energy

Back to the previous question. Let's use our new knife to cut the butter:

$$\frac{\partial (F - R)}{\partial f_{ij}} = I - \frac{\partial R}{\partial f_{ij}}$$

Let's rewrite the matrix F:

$$F = U\Sigma V^T = UV^T V\Sigma V^T = RS \Rightarrow R = UV^T$$

Therefore:

$$\frac{\partial R}{\partial f_{ij}} = (\frac{\partial U}{\partial f_{ij}})^T V + U^T \frac{\partial V}{\partial f_{ij}}$$

As you know, we already know how to solve the $\frac{\partial U}{\partial f_{ij}}$ and $\frac{\partial V}{\partial f_{ij}}$, we just repeat the steps above and then we get the answer.

References

[KE20] Theodore Kim and David Eberle. Dynamic deformables: implementation and production practicalities. In *ACM SIGGRAPH 2020 Courses*, pages 1–182. 2020.

[PL00] Théodore Papadopoulo and Manolis IA Lourakis. Estimating the jacobian of the singular value decomposition: Theory and applications. In *Computer Vision-ECCV 2000: 6th European Conference on Computer Vision Dublin, Ireland, June 26–July 1, 2000 Proceedings, Part I 6*, pages 554–570. Springer, 2000.