Geometry Mapping for Different Volumetric Elements

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1 Important Concepts

Basis Function

Consider a convex simplex, one can always expect to have a polynomial combination to represent the point inside it.

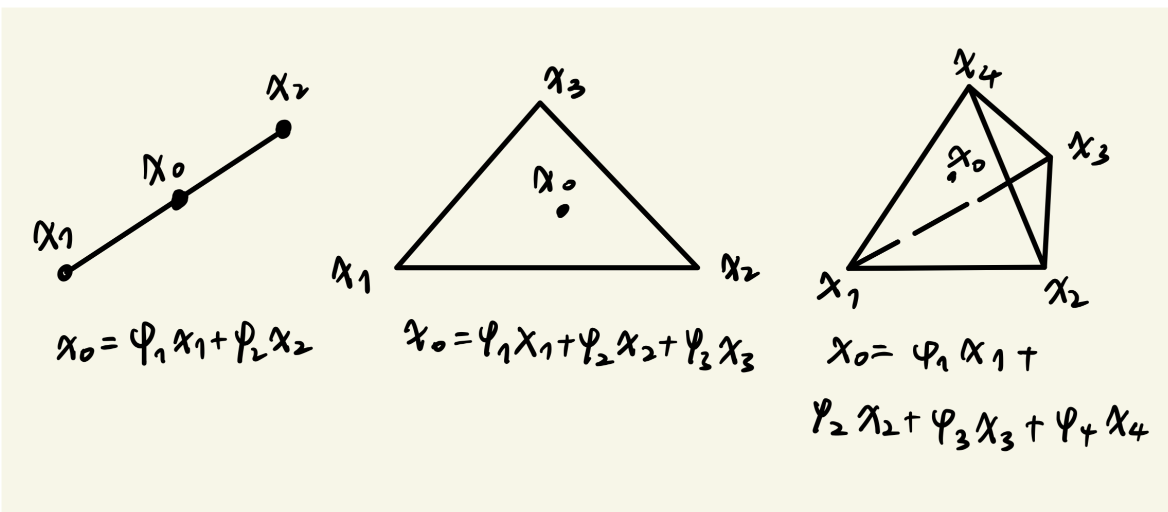

Figure 1: Linear Combination

In this case, for any point inside the convex simplex, we have:

$$X = \sum_{i=0}^{d} \phi_i(X) x_i$$

We usually call this the basis function.

Displacement Function

The displacement function is used to record the offset placed on the deformed element. Therefore, give the original x and the deformed x', the displacement is:

$$X_i + u(X_i) = x_i$$

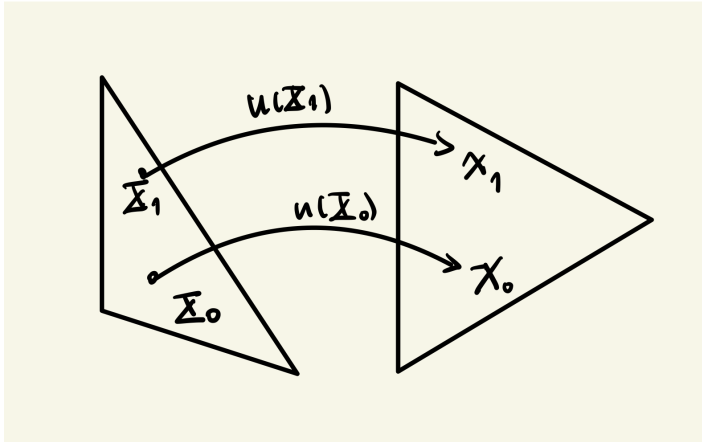

Figure 2: Displacement Function

To get the new position after mapping, we are interested in the interpolation of the displacement. That is, for a triangle, if we know the coefficients of the three vertices, then we can get the new position by interpolating, which can be written as(when we use the linear interpolation):

$$u(X) = \phi_0(X)u_0 + \phi_1(X)u_1 + \phi_2(X)u_2$$

We can also write it as:

$$u(X) = \sum_{i=0}^{d} \phi_i(X)u_i$$

General Form for the Deformation Jacobian

For now, we can combine the two points mentioned earlier together. Let's consider a ΔX change in the original element, which corresponds to a Δx in the deformed element.

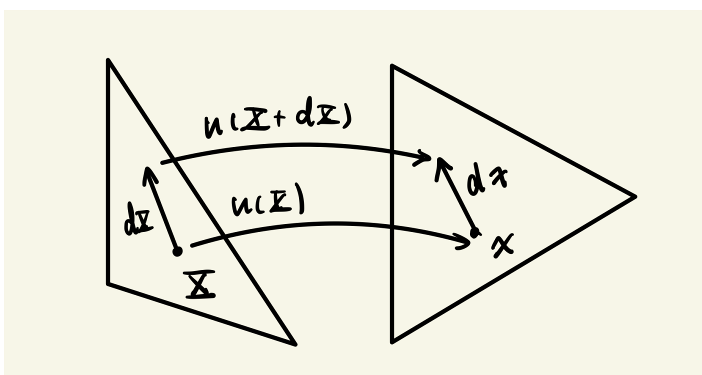

Figure 3: Relationship between the original and deformed element

In this drawing, we will have two equations:

$$\begin{cases} X + dX + u(X + dX) = x + dx \\ X + u(X) = x \end{cases}$$

We first do the Taylor expansion, and then plug the second equation into the first one, we got:

$$
\begin{aligned}
X + dX + u(X + dX) &= x + dx \\
X + dX + u(X) + \frac{\partial u}{\partial X} dX &= x + dx \\
\cancel{X} + dX + \cancel{u(X)} + \frac{\partial u}{\partial X} dX &= \cancel{x} + dx \\
I + \frac{\partial u}{\partial X} &= \frac{dx}{dX}
\end{aligned}
$$

We call $\frac{dx}{dX}$ as the Deformation Jacobian.

2 Basic Element

2.1 Triangle Element

Step 1: Build Basis and Displacement Functions

In the first step, we need to build the basis function and displacement function.

Let's consider two dimensions and write the point $P = (X, Y)$ instead of using a single X. For the basis function, we have:

$$
P = \phi_0(P)P_0 + \phi_1(P)P_1 + \phi_2(P)P_2 = \begin{bmatrix} \phi_0(P)X_0 + \phi_1(P)X_1 + \phi_2(P)X_2 \\ \phi_0(P)Y_0 + \phi_1(P)Y_1 + \phi_2(P)Y_2 \end{bmatrix}
$$

Therefore, the displacement function is:

$$
U(P) = \phi_0(P)U_0 + \phi_1(P)U_1 + \phi_2(P)U_2 = \begin{bmatrix} \phi_0(P)u_0 + \phi_1(P)u_1 + \phi_2(P)u_2 \\ \phi_0(P)v_0 + \phi_1(P)v_1 + \phi_2(P)v_2 \end{bmatrix}
$$

Step 2: Compute the Basis Function

We can use the Vandemon to build the linear system to solve the ϕ_i. Since the linear combination is $c_{i0} + c_{i1}X + c_{i2}Y$.(NOTE, this formula has no xy term since the triangle is singular linear) Take the ϕ_0 as an example:

$$
\begin{cases}
c_{00} + c_{01}X_0 + c_{02}Y_0 = 1 \\
c_{00} + c_{01}X_1 + c_{02}Y_1 = 0 \\
c_{00} + c_{01}X_2 + c_{02}Y_2 = 0
\end{cases}
$$

Solve this linear system and the ϕ_0 is:

$$
\phi_0(P) = \begin{bmatrix} c_{00} & c_{01} & c_{02} \end{bmatrix} \begin{bmatrix} 1 \\ X \\ Y \end{bmatrix} = c_{00} + c_{01}X + c_{02}Y
$$

To make the whole process easier, people usually use the reference element and set the normalized parameters. We can imagine a simple triangle as below, a simple triangle with three points $x_0 = (0, 0), x_1 = (1, 0).x_2 = (0, 1)$:

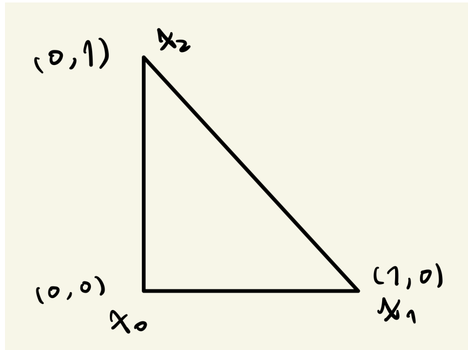

Figure 4: Reference Triangle

By solving the linear system, we will get the following:

$$\begin{cases} \phi_0(P) = 1 - X - Y \\ \phi_1(P) = X \\ \phi_2(P) = Y \end{cases}$$

Of Course, we can make this more general. We can build a new coordinate system. Here is how it works:

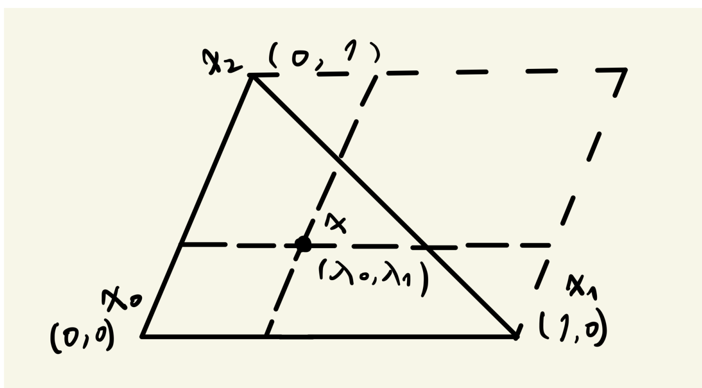

Figure 5: New Aligned Coordinate System

As you can see, we can encode the position as $P = P(\lambda_0, \lambda_1)$, which only records the position ratio. For now, we can rewrite the basis function above as:

$$\begin{cases} \phi_0(P) = 1 - \lambda_0 - \lambda_1 \\ \phi_1(P) = \lambda_0 \\ \phi_2(P) = \lambda_1 \end{cases}$$

Where $\lambda_0 \in [0,1], \lambda_1 \in [0,1]$.
Then we can write the original coordinates, basis function, and the replacement function as:

$$\begin{aligned} P &= P(\lambda_0, \lambda_1) \\ P &= (1 - \lambda_0 - \lambda_1)P_0 + \lambda_0 P_1 + \lambda_1 P_2 \\ U(P) &= (1 - \lambda_0 - \lambda_1)U_0 + \lambda_0 U_1 + \lambda_1 U_2 \end{aligned}$$

Where X_i and u_i are both $\in \Re^{2\times 1}$.

Step 3: Compute the Jacobian of Deformation
For now, we want to evaluate $\frac{dx}{dX}$ and therefore we need to compute the $\frac{\partial u}{\partial X}$. From the chain rule, we have:

$$\frac{\partial u}{\partial X} = \frac{\partial u}{\partial \lambda_0}\frac{\partial \lambda_0}{\partial X} + \frac{\partial u}{\partial \lambda_1}\frac{\partial \lambda_1}{\partial X}$$

From the Inverse Function Theorem ($y'(x) = \frac{1}{(x^{-1})'(y)}$):

$$\frac{\partial \lambda_i}{\partial X} = (\frac{\partial X}{\partial \lambda_i})^{-1}$$

More specifically, in the matrix, there is a formula:

$$\begin{bmatrix} \frac{\partial \lambda_0}{\partial X} & \frac{\partial \lambda_0}{\partial Y} \\ \frac{\partial \lambda_1}{\partial X} & \frac{\partial \lambda_1}{\partial Y} \end{bmatrix} = \begin{bmatrix} \frac{\partial X}{\partial \lambda_0} & \frac{\partial X}{\partial \lambda_1} \\ \frac{\partial Y}{\partial \lambda_0} & \frac{\partial Y}{\partial \lambda_1} \end{bmatrix}^{-1}$$

And then we need to compute:

$$\frac{\partial u}{\partial X} = \frac{\partial u}{\partial \lambda_0}(\frac{\partial X}{\partial \lambda_0})^{-1} + \frac{\partial u}{\partial \lambda_1}(\frac{\partial X}{\partial \lambda_1})^{-1}$$

Our final goal is $F = I + \frac{\partial u}{\partial X}$ and therefore, we need to compute

$$\begin{aligned} \frac{\partial U}{\partial P} &= \begin{bmatrix} \frac{\partial u}{\partial X} & \frac{\partial u}{\partial Y} \\ \frac{\partial v}{\partial X} & \frac{\partial v}{\partial Y} \end{bmatrix} = \begin{bmatrix} \frac{\partial U}{\partial \lambda_0} & \frac{\partial U}{\partial \lambda_1} \end{bmatrix} \begin{bmatrix} \frac{\partial P}{\partial \lambda_0} & \frac{\partial P}{\partial \lambda_1} \end{bmatrix}^{-1} \\ F &= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \frac{\partial U}{\partial P} \\ &= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} -U_0 + U_1 & -U_0 + U_2 \end{bmatrix} \begin{bmatrix} -P_0 + P_1 & -P_0 + P_2 \end{bmatrix}^{-1} \\ &= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} -u_0 + u_1 & -u_0 + u_2 \\ -v_0 + v_1 & -v_0 + v_2 \end{bmatrix} \begin{bmatrix} -X_0 + X_1 & -X_0 + X_2 \\ -Y_0 + Y_1 & -Y_0 + Y_2 \end{bmatrix}^{-1} \end{aligned}$$

High Order Basis
As we can see in the previous section, the final result actually are the same expect the gradient of the basis function. We can list several polynomial functions in 2D here:

$$\begin{cases} c_0 + c_1x + c_2y & \textbf{order 1} \\ c_0 + c_1x + c_2y + c_3xy + c_4x^2 + c_5y^2 & \textbf{order 2} \\ c_0 + c_1x + c_2y + c_3xy + c_4x^2 + c_5y^2 + c_6x^3 + c_7x^2y + c_8xy^2 + c_9y^3 & \textbf{order 3} \end{cases}$$

Therefore, it is obvious that people need to sample 6 nodes with order 2 and 10 nodes with order 3 function.

2.2 Tetrahedron

Step 1: Build Basis and Displacement Functions
Since there are four vertices in a 3D tetrahedron, then the Coordinates, Basis, and Displacement Functions should be like this:

$$\begin{aligned} P &= P(\lambda_1, \lambda_2, \lambda_3) \\ P &= \phi_0(\lambda_1, \lambda_2, \lambda_3)P_0 + \phi_1(\lambda_1, \lambda_2, \lambda_3)P_1 + \phi_2(\lambda_1, \lambda_2, \lambda_3)P_2 + \phi_3(\lambda_1, \lambda_2, \lambda_3)P_3 \\ U &= \phi_0(\lambda_1, \lambda_2, \lambda_3)U_0 + \phi_1(\lambda_1, \lambda_2, \lambda_3)U_1 + \phi_2(\lambda_1, \lambda_2, \lambda_3)U_2 + \phi_3(\lambda_1, \lambda_2, \lambda_3)U_3 \end{aligned}$$

Where P_i and U_i are both tensor $\in \Re^{3\times1}$.

$$P_i = \begin{bmatrix} X_i \\ Y_i \\ Z_i \end{bmatrix}, U_i = \begin{bmatrix} u_i \\ v_i \\ w_i \end{bmatrix}$$

Step 2: Compute the Basis Function

We only need to change the ϕ_i's polynomial terms for the different order. If we use the linear basis function. then we have:

$$\begin{cases} \phi_0(\lambda_1,\lambda_2,\lambda_3) = 1-\lambda_1-\lambda_2-\lambda_3 \\ \phi_1(\lambda_1,\lambda_2,\lambda_3) = \lambda_1 \\ \phi_2(\lambda_1,\lambda_2,\lambda_3) = \lambda_1 \\ \phi_3(\lambda_1,\lambda_2,\lambda_3) = \lambda_3 \end{cases}$$

One can expect to compute the higher-order polynomial formula from the reference tetrahedron element. The core idea is to use a new aligned coordinate system.

Step 3: Compute the Jacobian of Deformation

We need to compute the F matrix, which is:

$$\begin{aligned} F &= I + \frac{\partial u}{\partial X} \\ &= I + \frac{\partial u}{\partial \lambda}(\frac{\partial X}{\partial \lambda})^{-1} \\ &= I + \frac{\partial u}{\partial \lambda_1}(\frac{\partial X}{\partial \lambda_1})^{-1} + \frac{\partial u}{\partial \lambda_2}(\frac{\partial X}{\partial \lambda_2})^{-1} + \frac{\partial u}{\partial \lambda_3}(\frac{\partial X}{\partial \lambda_3})^{-1} \end{aligned}$$

With a simple rearrangement, we have:

$$F = \begin{bmatrix} \frac{x}{X} & \frac{x}{Y} & \frac{x}{Z} \\ \frac{y}{X} & \frac{y}{Y} & \frac{y}{Z} \\ \frac{z}{X} & \frac{z}{Y} & \frac{z}{Z} \end{bmatrix} = I + \begin{bmatrix} \frac{u}{X} & \frac{u}{Y} & \frac{u}{Z} \\ \frac{v}{X} & \frac{v}{Y} & \frac{v}{Z} \\ \frac{w}{X} & \frac{w}{Y} & \frac{w}{Z} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} + \begin{bmatrix} \frac{u}{\lambda_1} & \frac{u}{\lambda_2} & \frac{u}{\lambda_3} \\ \frac{v}{\lambda_1} & \frac{v}{\lambda_2} & \frac{v}{\lambda_3} \\ \frac{w}{\lambda_1} & \frac{w}{\lambda_2} & \frac{w}{\lambda_3} \end{bmatrix} \begin{bmatrix} \frac{X}{\lambda_1} & \frac{X}{\lambda_2} & \frac{X}{\lambda_3} \\ \frac{Y}{\lambda_1} & \frac{Y}{\lambda_2} & \frac{Y}{\lambda_3} \\ \frac{Z}{\lambda_1} & \frac{Z}{\lambda_2} & \frac{Z}{\lambda_3} \end{bmatrix}^{-1}$$

2.3 Thick Shell Model

The thick shell model has many different models. The key for the thick shell is that it should take the thickness and the distortion between the shell into consideration.

Step 1: Build Basis and Displacement Functions

For the triangular prism element, we can treat it as a combination of three normals and a mid-surface triangle.

$$\begin{aligned} P &= P(\lambda_1,\lambda_2,\lambda_3) \\ P &= \phi_3(\lambda_3)(\phi_0(\lambda_1,\lambda_2)P_1 + \phi_1(\lambda_1,\lambda_2)P_2 + \phi_2(\lambda_1,\lambda_2)P_3) \\ &\quad + \phi_4(\lambda_3)(\phi_0(\lambda_1,\lambda_2)P_4 + \phi_1(\lambda_1,\lambda_2)P_5 + \phi_2(\lambda_1,\lambda_2)P_6) \\ U &= \phi_3(\lambda_3)(\phi_0(\lambda_1,\lambda_2)U_1 + \phi_1(\lambda_1,\lambda_2)U_2 + \phi_2(\lambda_1,\lambda_2)U_3) \\ &\quad + \phi_4(\lambda_3)(\phi_0(\lambda_1,\lambda_2)U_4 + \phi_1(\lambda_1,\lambda_2)U_5 + \phi_2(\lambda_1,\lambda_2)U_6) \end{aligned}$$

Have to mention, in this case, on the triangle side, it is able to up to any high order basis. However, the vertical sides are always linear.

Step 2: Compute the Basis Function

Since the triangle face side is a simple triangle basis function, people can directly adapt the triangle basis function into the formula above.

Step 3: Compute the Jacobian of Deformation

This is similar to the triangle and tetrahedron.